Abstract

Stress singularity frequently occurs at a vertex in the interface of joints due to a discontinuity of materials. Stress singularity fields are one of the main factors responsible for debonding under mechanical or thermal loadings. Stress distribution near the vertex in the interface of joints is very important for maintaining the reliability of joints. However, the intensity of singularity for 3D transversely isotropic piezoelectric single-step bonded joints has not been made clear until now. In this paper, the intensity of singularity in transversely isotropic piezoelectric dissimilar material joints is analyzed. The orders of singularity at a vertex and at a point on singularity lines for piezoelectric bonded joints are determined using the Eigen analysis. The distributions for stress and electric displacement on the interface and the intensities of singularity for stress and electric displacement are investigated using BEM. From the numerical results, it is shown that the intensity of singularity increases with the increase of material thickness in the joint.

Keywords: Boundary element method, Piezoelectric dissimilar material, Smart structure stress singularity, Transversely isotropic bonded joint.

1. Introduction

In recent years, intelligent or smart structures and systems have drawn more and more attention. Piezoelectric materials have been extensively used as transducers and sensors due to the intrinsic direct and converse piezoelectric effects that take place between the electric field and mechanical deformation. Piezoelectric materials play a key role as active components in many engineering and technology fields such as electronics, laser, microwave infrared, navigation and biology [1997]. Mechanical stress occurs in piezoelectric material for any electric input. The stress concentrations caused by mechanical or electric loads may lead to crack initiation and extension, and sometimes the stress concentrations may be high enough to debond the material parts. Industrial products such as electronic devices and heat endurance parts are composed of dissimilar materials. A mismatch of material properties causes a failure at the free edge of joints because a stress concentration occurs along the free edge of the interface, especially at the joint’s vertex [2009]. When two materials are joined, a free-edge stress singularity usually develops at the intersection of the interface and free surface. Stress singularity is related to debonding and delamination.

Analysis of Intensity of Singular Stress Fields in Two-phase Transversely Isotropic Piezoelectric Single-step Bonded Joints

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at the interface of bonded joints. Several studies have investigated the stress singularity field in 3D elastic materials. Yamada and Okumura [1981] developed a finite element analysis for solving the Eigen value equation to directly determine the order of stress singularity and the angular variation of stress and displacement fields. Pageau et al. [1995] then adapted the Eigen analysis based on a finite element formulation to analyze the in-plane deformation of wedges and junctions of anisotropic materials. The stress and displacement fields were obtained using the Eigen formulation for real and complex orders of stress singularity. Pruksilailert and Koguchi [2005] examined the order of stress singularity not only at a vertex but also along the stress singularity line between two isotropic materials in joints using the Eigen analysis.

Stress distributions around the vertex were determined using a boundary element method (BEM). Koguchi [2006] determined the intensity of singularity by fitting the stress profile obtained from the BEM analysis with a least squares method. Ding et al. [1996] systematically studied the general solutions of equations for transversely isotropic piezoelectric materials and obtained the solutions of a half-space. Lee and Jiang [1994] obtained the boundary integral equation and two-dimensional solution by using the double Fourier transform and by considering one case of the Eigen values. Recently, Ikeda et al. [2011] proposed the solution of singular stress field and its SIFs of an interfacial corner of a 2D dissimilar piezoelectric material joints using an extended Stroh formalism. At present, no clear picture exists of the problem of intensity of singularity for 3D two-phase transversely isotropic piezoelectric dissimilar material joints. Therefore, the purpose of this study is to analyze the intensity of singularity in transversely isotropic piezoelectric dissimilar material joints.

2. The Basic formula

In the absence of body forces and free charges, the governing equations of three-dimensional piezoelectric materials are expressed as follows [1996]:

\[
\sigma_{ij} = 0, \quad D_i = 0
\]  

(1)

where \( \sigma_{ij} \) and \( D_i \) are the stress tensor and electric displacement vector, respectively. These equations are the elastic equilibrium equations and Gauss’s law of electrostatics, respectively. The constitutive relations for piezoelectric material are expressed as follows:

\[
\sigma_{ij} = c_{ijkl} e_{kl} - e_{ik} E_k, \quad D_i = e_{ik} E_k + \chi_{ik} \phi_k
\]  

(2)

where \( e_{ik} \) is the strain tensor, which is the mechanical field variables, \( E_k \) is electric field, \( c_{ijkl} \) is the elastic constant, \( e_{ik} (e_{ik}) \) and \( \chi_{ik} \) are the piezoelectric constant and electric permittivity (dielectric constant), respectively. The elastic strain-displacement and electric field-potential equations are expressed as follows:

\[
E_i = \frac{1}{2} (u_{ij} + u_{ij}'), \quad E_i = -\phi_i
\]  

(3)

where \( u_i \) and \( \phi \) are the elastic displacement and electric potential, respectively.

According to Ding et al. [1997], the fundamental solutions of the governing differential equations for the transversely isotropic piezoelectric material are as follows:

\[
u = \sum_{j=1}^{3} \frac{\partial \psi_j}{\partial x} \psi_j, \quad w = \sum_{j=1}^{3} \frac{\partial \psi_j}{\partial z} \psi_j, \quad \phi = \sum_{j=1}^{3} \alpha_j \frac{\partial \psi_j}{\partial z}
\]  

(4)

where, \( s_1, s_2, \) and \( s_3 \) are the three roots of the characteristic equation, which is related to the following equation.

\[
as^6 - bs^4 + cs^2 - d = 0
\]  

(5)

where \( a = c_{44} (e_{11} + c_{33} \chi_{33}) \),

\[
b = c_{33} \left[ (e_{11} + c_{33} \chi_{33}) \right] + (e_{11} + c_{33} \chi_{33}) \left[ e_{33} (c_{13} + c_{13} \chi_{33}) - (e_{11} + c_{33} \chi_{33}) \left[ 2c_{13} (e_{13} + e_{13} \chi_{33}) - 2(e_{11} + c_{33} \chi_{33}) (e_{13} + e_{13} \chi_{33}) \right] \right]
\]

\[
c = c_{33} \left[ (e_{11} + c_{33} \chi_{33}) \right] + (e_{11} + c_{33} \chi_{33}) \left[ e_{33} (c_{13} + c_{13} \chi_{33}) - (e_{11} + c_{33} \chi_{33}) \left[ 2c_{13} (e_{13} + e_{13} \chi_{33}) - 2(e_{11} + c_{33} \chi_{33}) (e_{13} + e_{13} \chi_{33}) \right] \right]
\]

\[
d = c_{33} \left[ (e_{11} + c_{33} \chi_{33}) \right]
\]

In three roots of Eq. (5), \( s_j \) is assumed to be a positive real number, \( s_1, s_2, \) and \( s_3 \) are either positive real numbers.
or a pair of conjugate complex roots with positive real parts. Function $\psi_i$ in Eq. (4) for an infinite piezoelectric material is given as follows:

$$
\psi_i = 0 \quad \text{and} \quad \psi_i = A_i \text{sign}(z - z_0) \ln \left( \frac{R_0 + s_i}{z - z_0} \right)
$$

where $i = 1, 2, 3$.

3. Boundary integral equation

Based on the Somigliana equation, the boundary integral formulation is expressed as follows:

$$
C(d\mathbf{u}(d)) = \int_\Gamma \mathbf{U}^t(d, x) \mathbf{t}(x) d\Gamma - \int_\Gamma \mathbf{T}^t(d, x) \mathbf{u}(x) d\Gamma
$$

where $C$ is the coefficient matrix which depends on the shape of the boundary $\Gamma$, and the general displacement vector $\mathbf{u}$, and surface traction vector $\mathbf{t}$ are as follows:

$$
\mathbf{u} = [u \quad v \quad w \quad -\phi]^T, \quad \mathbf{t} = [t_x \quad t_y \quad t_z \quad -\omega]^T
$$

and two matrices $\mathbf{U}^*$ and $\mathbf{T}^*$ composed of fundamental solutions are:

$$
\mathbf{U}^* = \begin{bmatrix}
    u_{x1} & u_{y1} & u_{z1} & \phi_{x1} \\
    u_{x2} & u_{y2} & u_{z2} & \phi_{x2} \\
    u_{x3} & u_{y3} & u_{z3} & \phi_{x3} \\
    u_{x4} & u_{y4} & u_{z4} & \phi_{x4}
\end{bmatrix},
\mathbf{T}^* = \begin{bmatrix}
    t_{x1} & t_{y1} & t_{z1} & a_{x1} \\
    t_{x2} & t_{y2} & t_{z2} & a_{x2} \\
    t_{x3} & t_{y3} & t_{z3} & a_{x3} \\
    t_{x4} & t_{y4} & t_{z4} & a_{x4}
\end{bmatrix}
$$

where $u_i^j$ and $t_i^j$ $(i, j = 1, 2, 3)$ are displacements and surface tractions, respectively at a field point $x$ in the $j$ coordinate direction due to a unit electric charge at $d$. $\phi_i^j$ and $\omega_i^j$ $(i = 1, 2, 3)$ are electric potentials and surface charges, respectively, at a field point $x$ due to a unit electric charge at $d$.

If the boundary is discretized with an eight-node isoparametric quadratic element, then the boundary integral equation is written as follows:

$$
C(d\mathbf{u}(d)) + \sum_{e=1}^{\sum} \int_{\Gamma_e} \mathbf{U}^t N_i |J| d\xi d\eta
$$

$$
= \sum_{e=1}^{\sum} \int_{\Gamma_e} \mathbf{T}^t N_i |J| d\xi d\eta
$$

where $N_i$ is the shape function, $J$ is a Jacobean matrix, $\mathbf{u}$ and $\mathbf{t}$ represent the displacement and surface traction at node, and $C$ is the coefficient matrix.

4. Eigen equation

An Eigen equation based on the finite element method (FEM) was used to analyze singularity at the singular point in a 3D dissimilar material joint. In the formulation of FEM, a spherical coordinate system with the origin at a singular point is introduced, and displacement within a sphere of radius $r_o$ in the singular field is expressed using the characteristic root $p$, which is related to the order of singularity. The surface of the sphere is divided into mesh. To determine the order of stress singularity, the Eigen equation was formulated as follows [2010]:

$$
\left[ (p^2 [A] + p [B] + [C]) \right] \{U\} = \{0\}
$$

where

$$
\{U\} = \{u_x \quad u_y \quad u_z \quad \psi\}, \quad [A] = \sum_{s} ([k_s - k_s]), \quad [B] = \sum_{s} ([k_s - k_s]), \quad [C] = \sum_{s} ([k_s - k_s])
$$

Here, $p$ represents the characteristic root, which is related to the order of singularity, $\lambda$, as $\lambda = 1 - p$. $[A]$, $[B]$ and $[C]$ are matrices composed of material properties, and $\{U\}$ represents the elastic displacement and electric potential vector.

5. Results and discussions

Figure 1 represents a model for 3D piezoelectric single-step bonded structure used in the present analysis. The dimensions of material 1 are 10x10x$h$ mm and those of material 2 are 20x20x5 mm. The displacement and electric potential in the $z$-direction on the bottom in the model is fixed. The model is subjected to uniform tension (1 MPA) and electric displacement (1 C/m$^2$) of which the poling direction is parallel to the $z$-axis. PZT-5A and PZT-7 are used for Materials 1 and 2, respectively, in the analysis.

Figure 2(a) represents the geometry of a typical case where singular stress occurs at point o. The region
surrounding the singular point is divided into a number of quadratic elements with a summit o, with each element being located in spherical coordinates r, $\theta$, and $\phi$ by its nodes 1 to 8. Figure 2(b) represents a mesh model for the 3D piezoelectric single-step bonded structure used in the present analysis. The mesh near the singular point is finer than the region farther from the singular point. The total numbers of elements and nodes are 4643 and 13931, respectively. The size of the smallest element is $2.73 \times 10^{-6}$ mm at singular point.

**Figure 1.** Model of analysis for two-phase piezoelectric bonded joint

![Figure 1](image1.png)

**Figure 2.** (a) Element geometry, and (b) a mesh model on x, y and z coordinates

![Figure 2](image2.png)

**TABLE 1: Material properties of piezoelectric materials**

<table>
<thead>
<tr>
<th>Material</th>
<th>Elastic Constant, $10^9$ N/m$^2$</th>
<th>Piezoelectric Constant, C/m$^2$</th>
<th>Dielectric Constant, $10^{-10}$ C/Vm</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$c_{11}$ $c_{12}$ $c_{13}$ $c_{33}$ $c_{44}$</td>
<td>$e_{31}$ $e_{15}$ $e_{33}$</td>
<td>$\chi_{11}$ $\chi_{33}$</td>
</tr>
<tr>
<td>PZT-5A</td>
<td>12.1 7.54 7.52 11.3 2.11</td>
<td>-5.20 12.3 15.8</td>
<td>81.1 73.5</td>
</tr>
<tr>
<td>PZT-7</td>
<td>13.0 8.30 8.30 11.9 2.50</td>
<td>-10.3 13.5 14.7</td>
<td>171 186</td>
</tr>
</tbody>
</table>
The order of singularity \( \lambda \), at the vertex for the above model is calculated using the Eigen analysis method. Solving the Eigen equation yields many roots \( p \) and Eigen vectors are obtained. However, if the root \( p \) is within the range of \( 0 < p < 1 \), this fact indicates that the stress field has singularity. The values of the order of singularity at the singularity corner are 0.5651, 0.2934, 0.1777, and 0.0958.

The distributions of stress and electric displacement in the singular field for the two-phase piezoelectric single-step bonded structure are obtained using the boundary element method. The distributions of stress and electric displacement in the singular field against radial distance, \( r \) for various thicknesses of upper material are shown in **Figures 3 to 6**. It is clear from the figures that stress and electric displacement increase with the thickness of the upper material. All figures show that stress and electric displacement are larger at the interface’s vertex. Therefore, there is a possibility of debonding and delamination at the corner of the bonded joint.

Stress distribution at the vertex in a singular field is expressed for the spherical coordinate system as follows [2010]:

\[
\sigma_{ij}(r, \theta, \phi) = K_{ij} f_{ij}(\theta, \phi) r^{-\lambda_{\text{vertex}}} + K_{ij} f_{ij}(\theta, \phi)
\]

where \( r \) is the distance from the vertex, \( \lambda_{\text{vertex}} \) is the order of stress singularity, \( K_{ij} (k = 1, 2) \) is the intensity of stress singularity, \( f_{ij}(\theta, \phi) (k = 1, 2) \) is the angular function for stress component, \( \sigma_{ij} \). Similarly, the electric displacement distribution at the vertex in a singular
field is expressed for the spherical coordinate system as follows:

\[ D_i (r, \theta, \phi) = M_{i1} l_{i1} (\theta, \phi) r^{-k_{\text{min}}} + M_{i2} l_{i2} (\theta, \phi) \]  

(11)

where \( M_{i} \) \((k = 1, 2)\) is the intensity of electric displacement singularity, and \( l_{ij} (\theta, \phi) \)(\(k = 1, 2\)) is the angular function for electric displacement component, \( D_i \).

In the present analysis, the intensities of singularity for stress and electric displacement are determined by considering the mechanical field effect that is larger than that of the electric field. The intensities of singularity for stress and electric displacement are calculated by fitting the stress and electric displacement curve with the help of the result of the Eigen analysis at the vertex. \textit{Figures 7 to 9} represent the relationship between the intensities of singularity for stress and the thickness of upper material. \textit{Figure 10} represents the relationship between the intensity of singularity for electric displacement and the thickness of the upper material. The intensity of singularity for both stress and electric displacement increases with the thickness of the upper material. This is due to the effect of the mechanical field that is larger than that of the electric field.

It is found from the present analysis that the value of \( K_{100} \) is larger than the value of \( K_{1r} \) and \( K_{1\phi} \). In the present analysis, it is also observed that the ratio of \( K_{100} \) and \( K_{1r} \) and the ratio of \( K_{1\theta} \) and \( K_{1\phi} \) with respect to the upper material’s thickness are nearly constant. There is a greater possibility of debonding and delamination at the corner of thick piezoelectric single-step bonded joints, because the effect of the intensities of singularity on a thick material joint is greater than that of a thin material.
Conclusion

In this paper, the order of singularity and the intensity of singularity that characterize a singular field at a vertex in 3D single-step dissimilar material joints were investigated using the Eigen analysis based on a finite element method and a boundary element analysis. The order of singularity at the vertex was calculated in accordance with the Eigen analysis.

The distributions of stress and electric displacement with respect to radial distance were calculated in boundary element analysis. Stress and electric displacement had a larger value at the vertex of the joint. The intensities of singularity were calculated by fitting the stress and electric displacement profile with the help of the results of the Eigen analysis at the vertex. The intensities of singularity for stress and electric displacement increase with the upper material’s thickness. Therefore, there is a greater possibility of debonding and delamination occurring at the corner of thick transversely isotropic piezoelectric single-step bonded joints.

References